# Electromagnetic Waves : Plane Waves in Non-Conducting Media

Maxwell's equations ⟶ provide us all the information that can be drawn from the classical theory of electric and magnetic fields.
Fields generated by the moving charges can leave the source and travel through the space in the form of waves. . One of the important features of Maxwell equations.
We re-write the Maxwell's eq$^n$s.

$$\nabla \cdot E = \rho/\varepsilon_0 \quad \text{or} \quad \nabla \cdot D = \rho \quad \text{---(1)}$$

$$\nabla \cdot B = 0, \quad \nabla \cdot H = 0 \quad \text{---(2)}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{---(3)}$$

$$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}, \quad \nabla \times H = J + \frac{\partial D}{\partial t} \quad \text{---(4)}$$

(a) In vacuum $D = \varepsilon_0 E$, $B = \mu_0 H$, $J = 0$, $\rho = 0$

So $\nabla \cdot E = 0$ ---(i) $\quad \nabla \cdot B = \nabla \cdot \mu_0 H = 0$ ---(ii)

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\mu_0 \frac{\partial H}{\partial t} \quad \text{---(iii)}$$

$$\nabla \times H = \varepsilon_0 \frac{\partial E}{\partial t} \quad \text{---(iv)}$$

(b) In any homogeneous medium for charge free case

$\nabla \cdot E = 0$ ---(i) $\quad \nabla \cdot H = 0$ ---(ii)

$$\nabla \times E = -\mu \frac{\partial H}{\partial t} \quad \text{---(iii)}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times H = \varepsilon \frac{\partial E}{\partial t}, \quad \nabla \times B = \mu \varepsilon \frac{\partial E}{\partial t} \quad \text{---(iv)}$$

Curl of eqⁿ (iii) gives.

$$\nabla \times (\nabla \times E) = -\nabla \times \frac{\partial B}{\partial t}$$

$$= -\mu \frac{\partial}{\partial t}(\nabla \times H)$$

Assume that $\varepsilon$, $\mu$ and $\sigma$ are constant.

Using vector identity on the left hand side and eqⁿ (iv) on R.H.S.

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\mu \frac{\partial J}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2}$$

i.e. $$\nabla^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} - \mu\sigma \frac{\partial E}{\partial t} = \nabla(\rho/\varepsilon_0) \quad \text{--- (1)}$$

In a region in which there is no free charge $\rho = 0$, therefore

$$\nabla^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} - \mu\sigma \frac{\partial E}{\partial t} = 0 \quad \text{--- (2)}$$

Assuming $E(r,t) = E(r)e^{-i\omega t}$, we can compare the relative magnitudes of the second and third terms in eqⁿ (2), Thus,

$$\frac{\left|\mu\sigma \frac{\partial E}{\partial t}\right|}{\left|\mu\varepsilon \frac{\partial^2 E}{\partial t^2}\right|} \simeq \frac{\sigma}{\varepsilon\omega} = \frac{1}{\omega\tau} = \frac{T}{2\pi\tau} \quad \text{--- (3)}$$

Where we have replaced $\frac{\varepsilon}{\sigma}$ by relaxation time $\tau$ and $\omega = \frac{2\pi}{T}$, $T$ being the period of oscillation. If $\tau << T$, the case for a conducting medium, the third term in eqⁿ (2) is dominant and we can write eqⁿ (2) as

$$\nabla^2 E - \mu\sigma \frac{\partial E}{\partial t} = 0 \quad \text{--- (4)}$$

a diffusion eqⁿ.

If $\tau >> T$, the term involving $\sigma$ can be neglected and we have eqⁿ for non-conducting medium

$$\boxed{\nabla^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0} \quad \text{---} (5)$$

exactly similar eq^n's can be obtained for H, viz

$$\nabla^2 H - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} - \mu\sigma \frac{\partial H}{\partial t} = 0 \quad \text{---} (6)$$

and

$$\boxed{\nabla^2 H - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} = 0} \quad \text{---} (7)$$

These eq^n's are of the type of wave eq^n's we are familiar with. The general wave-eq^n is

$$\nabla^2 \psi - \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad \text{---} (8)$$

where $v$ is the velocity of propagation.

Comparing eq^n (5) and (7) with eq^n (8) shows that the velocity in our case is

$$v = (\varepsilon\mu)^{-1/2} \quad \text{---} (9)$$

In vacuum $\quad v = (\varepsilon_0 \mu_0)^{-1/2} \quad \text{---} (10)$

We can conclude that any time variations in electric or magnetic field are propagated with the same velocity

$$v = (\varepsilon\mu)^{-1/2}$$

The simplest wave that is a sol^n of eq^n (5) and (7) is plane wave. Plane waves are good approximation to actual waves in many situations, we discuss the plane wave sol^n of above eq^n's.

Plane Wave → One in which the wave amplitude - the field vector component - is constant over all points of a plane normal to the direction of propagation. This plane constitutes a wave-front which advances with a velocity $v$ in a direction normal to itself. The field vector components that lie in given plane are functions of perpendicular distance of the plane from the origin and also of the time.

We may choose our coordinate system in such a way that the direction of propagation coincide with, say, x-axis. The wave ~~eqn~~ eqn takes one dimensional form such as

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0 \quad \text{---(11)}$$

This has a general solution

$$\psi(x,t) = A e^{i(kx-\omega t)} + B e^{-i(kx+\omega t)}$$

$$= f(x-vt) + g(x+vt) \quad \text{---(12)}$$

where $A, B \rightarrow$ are constants (generally complex)

and $k = \frac{\omega}{v}$, $v \rightarrow$ phase velocity of the wave.

Eqn (12) represents wave travelling to the right and to the left with velocity $v$.

We may assume, that the plane wave fields are of the form

$$E(x,t) = E_0 e^{i(kx-\omega t)} \quad \text{---(13)}$$

$$H(x,t) = H_0 e^{i(kx-\omega t)} \quad \text{---(14)}$$

We have assumed $E$ and $H$ to be in phase.

The justification as follows;